On relative t-designs in polynomial association schemes

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Abstract

Motivated by the similarities between the theory of spherical t-designs and that of t-designs in Q-polynomial association schemes, we study two versions of *relative t-designs*, the counterparts of Euclidean t-designs for P- and/or Q-polynomial association schemes. We develop the theory based on the *Terwilliger algebra*, which is a noncommutative associative semisimple $\mathbb{C}$-algebra associated with each vertex of an association scheme. We compute explicitly the Fisher type lower bounds on the sizes of relative t-designs, assuming that certain irreducible modules behave nicely. The two versions of relative t-designs turn out to be equivalent in the case of the Hamming schemes. From this point of view, we establish a new algebraic characterization of the Hamming schemes.

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1 Introduction

Design theory is concerned with finding "good" finite sets that "approximate globally" their underlying spaces (often) having strong symmetry/regularity, such as the Euclidean space $\mathbb{R}^n$, the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$, and the set of k-subsets of a given v-set. It has therefore a vast range of applications in various fields of science. See, e.g., [8, 2].

The similarities between the theories of spherical t-designs and combinatorial t-(v, k, λ) designs are well known; cf. [13, 12, 17, 1]. Historically, the concept of spherical t-designs was introduced by Delsarte, Goethals, and Seidel [13] as a continuous analogue of that of t-designs in Q-polynomial association schemes due to Delsarte [9, 10]. (Combinatorial t-(v, k, λ) designs are precisely the t-designs in the Johnson scheme $J(v, k)$.) It was then generalized to the concept of Euclidean t-designs by Neumaier and Seidel [20], and

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Euclidean t-designs quickly became an active area of research; cf. [2]. Although the counterparts of Euclidean t-designs in the theory of Q-polynomial association schemes were already defined and discussed to some extent by Delsarte [11] (cf. [3]) much earlier as *relative t-designs*, it seems that the theory of the latter has not been fully developed yet (except in the case of the binary Hamming scheme $H(n, 2)$, in which case relative t-designs turn out to be equivalent to *regular t-wise balanced designs*). This paper is a contribution to this theory. Our discussions also include a concept of relative t-designs in general P-polynomial association schemes as well, following Delsarte and Seidel [14].

We refer the reader to [9, 5, 6, 17, 19], etc., for the background on association schemes and some fundamental concepts. Throughout the paper, let $\mathfrak{X} = (X, \{R_r\}_{r=0}^d)$ be a d-class association scheme, and fix a base vertex $u_0 \in X$. Let $X_r = \{x \in X \mid (u_0, x) \in R_r\}$ for $r = 0, 1, \dots, d$. We call $X_0, X_1, \dots, X_d$ the *shells* of $\mathfrak{X}$. Let $\mathcal{F}(X)$ be the vector space consisting of all the real valued functions on X. In the following arguments we often identify $\mathcal{F}(X)$ with the vector space $\mathbb{R}^X$ consisting of the real column vectors with coordinates indexed by X.

We first introduce a concept of t-designs for general P-polynomial association schemes. Suppose that $\mathfrak{X}$ is P-polynomial. In the study of spherical/Euclidean t-designs in $\mathbb{R}^n$, we work with the vector space of polynomials in n variables, in particular with the subspaces of homogeneous polynomials. For the P-polynomial scheme $\mathfrak{X}$, it is natural to consider the following subspaces of $\mathcal{F}(X)$. For every $z \in X_j$, we define $f_z \in \mathcal{F}(X)$ by

$$f_z(x) = \begin{cases} 1, & \text{if } x \in X_i,\ i \geq j, \text{ and } (x, z) \in R_{i-j}, \\ 0, & \text{otherwise,} \end{cases} \qquad (x \in X).$$

In other words, $f_z(x) = 1$ if and only if z lies on a geodesic between u_0 and x in the corresponding distance-regular graph (X, R_1). Let $\mathrm{Hom}_j(X) = \mathrm{span}\{f_z \mid z \in X_j\}$ ($j = 0, 1, \dots, d$). Then,

$$\dim(\mathrm{Hom}_j(X)) = |X_j| =: k_j \quad (j = 0, 1, \dots, d),$$

and we have the following direct sum decomposition of $\mathcal{F}(X)$:

$$\mathcal{F}(X) = \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_d(X).$$

We now consider a (positive) weighted subset (Y, w) of X. Let $\{r_1, r_2, \dots, r_p\} = \{r \mid Y \cap X_r \neq \emptyset\}$, and let $Y_{r_i} = Y \cap X_{r_i}$, $w(Y_{r_i}) = \sum_{y \in Y_{r_i}} w(y)$ for $i = 1, 2, \dots, p$. We say that Y is *supported* by the union $S = X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$ of p shells. For any subspace $R(X)$ of $\mathcal{F}(X)$, we write $R(S) = \{f|_S \mid f \in R(X)\}$.

Definition 1.1 (P-polynomial case)**.** A weighted subset (Y, w) of X is a *relative t-design of $\mathfrak{X}$ with respect to u_0* if

$$\sum_{i=1}^{p} \frac{w(Y_{r_i})}{k_{r_i}} \sum_{x \in X_{r_i}} f(x) = \sum_{y \in Y} w(y) f(y)$$

for every $f \in \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_t(X)$.

This definition is due to Delsarte and Seidel [14, Section 6] for the binary Hamming scheme $H(d,2)$. In this paper, we mostly consider the case $t = 2e$ for simplicity.

Theorem 1.2 ([14])**.** *Let (Y,w) be a relative $2e$-design of a Hamming scheme $H(d,q)$ with respect to u_0 in the sense of Definition 1.1. Let $S = X_{r_1} \cup \cdots \cup X_{r_p}$ be the union of the shells which support Y. Then,*

$$|Y| \geq \dim \big(\mathrm{Hom}_0(S) + \mathrm{Hom}_1(S) + \cdots + \mathrm{Hom}_e(S)\big). \tag{1.1}$$

Delsarte and Seidel [14] proved Theorem 1.2 only for $H(d,2)$, but their proof works for general q. Theorem 1.2 also follows from Theorem 1.4 and Proposition 1.5 below. Recently, Xiang [29] succeeded in determining the right hand side of (1.1) explicitly for $H(d,2)$, which was left open in [14]. Namely, he proved

$$\dim \big(\mathrm{Hom}_0(S) + \mathrm{Hom}_1(S) + \cdots + \mathrm{Hom}_e(S)\big) = k_e + k_{e-1} + \cdots + k_{e-p+1}, \tag{1.2}$$

under a reasonable additional condition which avoids the triviality. In this paper, we focus on generalizing (1.2) to other classes of P-polynomial association schemes (without necessarily reference to Theorem 1.2 itself). In Appendix A, we do, however, show that Theorem 1.2 is valid for dual polar schemes as well.

The concept of relative t-designs for Q-polynomial association schemes was introduced by Delsarte [11]. We now recall the definition. Suppose that $\mathfrak{X}$ is Q-polynomial. Let $E_0, E_1, \ldots, E_d$ be the Q-polynomial ordering of the primitive idempotents of $\mathfrak{X}$, and let $L_j(X)(\subseteq \mathcal{F}(X))$ be the column space of E_j $(j = 0, 1, \ldots, d)$. Then,

$$\dim(L_j(X)) = \mathrm{rank}(E_j) =: m_j \quad (j = 0, 1, \ldots, d),$$

and we have the following orthogonal direct sum decomposition of $\mathcal{F}(X)$:

$$\mathcal{F}(X) = L_0(X) \perp L_1(X) \perp \cdots \perp L_d(X).$$

Definition 1.3 (Q-polynomial case)**.** A weighted subset (Y,w) of X is *a relative t-design of $\mathfrak{X}$ with respect to u_0* if

$$\sum_{i=1}^{p} \frac{w(Y_{r_i})}{k_{r_i}} \sum_{x \in X_{r_i}} f(x) = \sum_{y \in Y} w(y) f(y)$$

for every $f \in L_0(X) \perp L_1(X) \perp \cdots \perp L_t(X)$.

Bannai and Bannai [3] obtained the following Fisher type inequality for general Q-polynomial association schemes:

Theorem 1.4 ([3])**.** *Let (Y,w) be a relative $2e$-design of the Q-polynomial scheme $\mathfrak{X}$ with respect to u_0 in the sense of Definition 1.3. Let $S = X_{r_1} \cup \cdots \cup X_{r_p}$ be the union of the shells which support Y. Then,*

$$|Y| \geq \dim \big(L_0(S) + L_1(S) + \cdots + L_e(S)\big). \tag{1.3}$$

As in the case of (1.1), it was not easy to compute the right hand side of (1.3) explicitly. The initial attempt was made by Li, Bannai, and Bannai [18] for $H(d,2)$, but was unsuccessful in general. Then, this attempt lead Xiang to obtain a successful result in the general case for $H(d,2)$, as it is known that the two definitions of relative t-designs are essentially equivalent for $H(d,2)$. Namely, both definitions are shown to be equivalent to the geometric definition of relative t-designs coming from the structure of the regular semilattice associated with $H(d,2)$; cf. [11]. The equivalence of Definition 1.1 for $H(d,2)$ with the definition of regular t-wise balanced designs was pointed out by Delsarte and Seidel [14, Theorem 6.2], whereas the equivalence of Definition 1.3 for $H(d,2)$ with the geometric definition of relative t-designs was established by Delsarte [11, Theorem 9.8] (see also [4]). However, we note that

Proposition 1.5. *If $\mathfrak{X}$ is a Hamming scheme $H(d,q)$, then for $t=0,1,\dots,d$,*

$$\mathrm{Hom}_0(X)+\mathrm{Hom}_1(X)+\cdots+\mathrm{Hom}_t(X)=L_0(X)+L_1(X)+\cdots+L_t(X). \tag{1.4}$$

Proof. Without loss of generality, we may suppose that $X=\{0,1,\dots,q-1\}^d$ and $u_0=(0,0,\dots,0)$. Let $z=(z_1,z_2,\dots,z_d)\in X_j$. Note that z has exactly j nonzero entries, and let $\ell_1,\ell_2,\dots,\ell_j$ be the corresponding coordinates. Then, it is easy to see that f_z is the characteristic function of the subset $\{(x_1,x_2,\dots,x_d)\in X \mid x_{\ell_h}=z_{\ell_h}\ (h=1,2,\dots,j)\}$, which is known to be contained in $\sum_{i=0}^{j} L_i(X)$; see, e.g., [10, 21].[1] Since both sides of (1.4) have the same dimension, we obtain the desired result. □

Thus, for $H(d,q)$, relative t-designs in the sense of Definition 1.1 are equivalent to relative t-designs in the sense of Definition 1.3. This observation seems to be new for $H(d,q)$ for general q. As is mentioned before, for $H(d,2)$, the result of Xiang [29] implies that the right hand side of (1.3) is also given explicitly by

$$\dim\big(L_0(S)+L_1(S)+\cdots+L_e(S)\big)=m_e+m_{e-1}+\cdots+m_{e-p+1}, \tag{1.5}$$

since $m_j=k_j$ $(j=0,1,\dots,d)$ in this case. In a private communication, Xiang extended his main result in [29] to general q. Thus, the right hand side of (1.3) is also given explicitly as (1.5) for $H(d,q)$.

In this paper, we investigate to what extent the above results can be generalized to other P- and/or Q-polynomial association schemes. In Section 2, we derive sufficient conditions that (1.2) (resp. (1.5)) holds for a P-polynomial (resp. Q-polynomial) association scheme (Theorems 2.1 and 2.5). These conditions can be readily checked for $H(d,q)$, so that we obtain different proofs of the results of Xiang mentioned above. Concerning (1.4), we first suspected that a similar result might hold for general (formally) self-dual P- and Q-polynomial association schemes, but it turns out that this is not the case in general. Indeed, in Section 3, we show that if $\mathfrak{X}$ is formally self-dual, P-polynomial (and thus Q-polynomial), and satisfies $\mathrm{Hom}_0(X)+\mathrm{Hom}_1(X)=L_0(X)+L_1(X)$, then $\mathfrak{X}$ must be a Hamming scheme $H(d,q)$, provided that $d\geq 6$ (Theorem 3.2). All of these theorems are proved using the theory of the *Terwilliger algebra* [25, 26, 27]. See [23] for more applications of the Terwilliger algebra to design theory.

[1] In Appendix B, we give a direct proof that f_z belongs to $\sum_{i=0}^{j} L_i(X)$, which does not use the theory of regular semilattices found in [10, 21].

2 Computations of the Fisher type lower bounds

In this section and the next, we shall freely use elementary facts about the Terwilliger algebra found in [25]. In this context, we shall work with the complex vector space $\mathbb{C}^X$ instead of $\mathbb{R}^X$, but we note that the dimensions of the various subspaces in question do not change, as they are spanned by real vectors. We use the following notation. Let $A_0, A_1, \dots, A_d$ and $E_0, E_1, \dots, E_d$ be (fixed orderings of) the adjacency matrices and the primitive idempotents of $\mathfrak{X}$, respectively. Let $E_0^*, E_1^*, \dots, E_d^*$ and $A_0^*, A_1^* \dots, A_d^*$ be the diagonal matrices with diagonal entries $(E_i^*)_{xx} = (A_i)_{u_0x}$ and $(A_i^*)_{xx} = |X|(E_i)_{u_0x}$ ($x \in X$, $i = 0, 1, \dots, d$). They form two bases of the *dual Bose–Mesner algebra with respect to* u_0. When we assume that $\mathfrak{X}$ is P-polynomial (resp. Q-polynomial), we understand that $A_0, A_1, \dots, A_d$ (resp. $E_0, E_1, \dots, E_d$) is the P-polynomial ordering (resp. Q-polynomial ordering) and write $A = A_1 = \sum_{i=0}^d \theta_i E_i$ (resp. $A^* = A_1^* = \sum_{i=0}^d \theta_i^* E_i^*$). The *Terwilliger algebra* T is the subalgebra of the full matrix algebra generated by the Bose–Mesner algebra and the dual Bose–Mesner algebra. The *endpoint*, *dual endpoint*, and the *diameter* of an irreducible T-module W are defined by $\rho(W) = \min\{i \mid E_i^* W \neq 0\}$, $\rho^*(W) = \min\{i \mid E_i W \neq 0\}$, and $\delta(W) = |\{i \mid E_i^* W \neq 0\}| - 1$, respectively.[2] For every $x \in X$, let $\hat{x} \in \mathcal{F}(X)$ be the characteristic function of the set $\{x\}$. We note that if $\mathfrak{X}$ is P-polynomial then $\mathrm{Hom}_j(X) = \mathrm{span}\bigl\{ \bigl(\sum_{i=j}^d E_i^* A_{i-j} E_j^* \bigr) \hat{z} \bigm| z \in X_j \bigr\}$ ($j = 0, 1, \dots, d$).

Theorem 2.1. *Suppose that $\mathfrak{X}$ is P-polynomial, and let $e, r_1, r_2, \dots, r_p$ be integers with $p - 1 \leq e \leq r_1 < r_2 < \cdots < r_p \leq d$. Suppose that every irreducible T-module W with endpoint at most e is thin and satisfies $\rho(W) + \delta(W) \geq r_p$. If the $p \times p$ matrix*

$$\begin{pmatrix} 1 & c_{r_1-e+p-1} & \cdots & (c_{r_1-e+p-1} \cdots c_{r_1-e+1}) \\ \vdots & \vdots & & \vdots \\ 1 & c_{r_p-e+p-1} & \cdots & (c_{r_p-e+p-1} \cdots c_{r_p-e+1}) \end{pmatrix} \tag{2.1}$$

consisting of the intersection numbers $c_i = p^i_{1,i-1}$ ($i = 1, 2, \dots, d$) is nonsingular, then

$$\dim \bigl(\mathrm{Hom}_0(S) + \mathrm{Hom}_1(S) + \cdots + \mathrm{Hom}_e(S) \bigr) = k_e + k_{e-1} + \cdots + k_{e-p+1},$$

where $S = X_{r_1} \cup X_{r_2} \cup \cdots \cup X_{r_p}$.

Proof. Fix a set $\mathcal{W}$ of irreducible T-modules in $\mathbb{C}^X$ such that $\mathbb{C}^X = \bigoplus_{W \in \mathcal{W}} W$. Observe that

$$\mathbb{C}^{X_j} = E_j^* \mathbb{C}^X = \bigoplus_{\substack{W \in \mathcal{W} \\ \rho(W) \leq j}} E_j^* W \quad (j = 0, 1, \dots, d),$$

so that

$$\mathrm{Hom}_j(S) = \bigoplus_{\substack{W \in \mathcal{W} \\ \rho(W) \leq j}} \left(\sum_{i=1}^p E_{r_i}^* A_{r_i - j} E_j^* \right) E_j^* W \quad (j = 0, 1, \dots, e).$$

[2] In [25, 26, 27], $\rho(W)$, $\rho^*(W)$, and $\delta(W)$ are called the dual endpoint, endpoint, and the dual diameter of W, respectively.

In particular, it follows that

$$\left(\sum_{j=0}^{e}\mathrm{Hom}_j(S)\right)\cap W=\sum_{j=\rho(W)}^{e}\left(\sum_{i=1}^{p}E^*_{r_i}A_{r_i-j}E^*_j\right)E^*_jW\subseteq\bigoplus_{i=1}^{p}E^*_{r_i}W$$

for every $W\in\mathcal{W}$ with $\rho(W)\leq e$, and that

$$\sum_{j=0}^{e}\mathrm{Hom}_j(S)=\bigoplus_{\substack{W\in\mathcal{W}\\ \rho(W)\leq e}}\left(\sum_{j=0}^{e}\mathrm{Hom}_j(S)\right)\cap W.$$

Pick any $W\in\mathcal{W}$ with $\rho:=\rho(W)\leq e$, and let v be a nonzero vector in $E^*_\rho W$. First, suppose that $\rho\leq e-p+1$. Since W is thin, for $j=e-p+1,\dots,e$, the vector $v_j=E^*_jA^{j-\rho}v$ is nonzero and hence is a basis of E^*_jW. Moreover, it follows that

$$\begin{aligned}\left(\sum_{i=1}^{p}E^*_{r_i}A_{r_i-j}E^*_j\right)v_j&=\sum_{i=1}^{p}\frac{1}{c_{r_i-j}\dots c_2c_1}E^*_{r_i}A^{r_i-j}E^*_jv_j\\&=\sum_{i=1}^{p}\frac{1}{c_{r_i-j}\dots c_2c_1}E^*_{r_i}A^{r_i-\rho}v\\&=\sum_{i=1}^{p}\frac{1}{c_{r_i-j}\dots c_2c_1}E^*_{r_i}A^{r_i-e+p-1}E^*_{e-p+1}A^{e-p+1-\rho}v\\&=\sum_{i=1}^{p}(c_{r_i-e+p-1}\dots c_{r_i-j+1})E^*_{r_i}A_{r_i-e+p-1}v_{e-p+1}.\end{aligned}$$

Since the coefficient matrix (2.1) is nonsingular, the vectors $\left(\sum_{i=1}^{p}E^*_{r_i}A_{r_i-j}E^*_j\right)v_j$ ($j=e-p+1,\dots,e$) are linearly independent. Thus, $\dim\left(\left(\sum_{j=0}^{e}\mathrm{Hom}_j(S)\right)\cap W\right)=p$. Next, suppose that $e-p+2\leq\rho\leq e$. Likewise, we find that the vectors $\left(\sum_{i=1}^{p}E^*_{r_i}A_{r_i-j}E^*_j\right)v_j$ ($j=\rho,\dots,e$) are linearly independent, and that $\dim\left(\left(\sum_{j=0}^{e}\mathrm{Hom}_j(S)\right)\cap W\right)=e-\rho+1$. Thus, it follows that

$$\begin{aligned}\dim\left(\sum_{j=0}^{e}\mathrm{Hom}_j(S)\right)&=\sum_{\substack{W\in\mathcal{W}\\ \rho(W)\leq e}}\min\{p,e-\rho(W)+1\}\\&=\sum_{\substack{W\in\mathcal{W}\\ \rho(W)\leq e}}\sum_{j=e-p+1}^{e}\dim(E^*_jW)\\&=\sum_{j=e-p+1}^{e}\dim(E^*_j\mathbb{C}^X)\\&=\sum_{j=e-p+1}^{e}k_j,\end{aligned}$$

as desired. □

Remark 2.2. In view of [7, Lemma 5.1], the assumption $\rho(W)+\delta(W)\geq r_p$ in Theorem 2.1 holds provided that $r_p\leq d-e$.

Example 2.3. Suppose that $\mathfrak{X}$ is a Hamming scheme $H(d,q)$. Then, $c_i=i$ ($i=1,2,\dots,d$). Thus, it follows that the matrix (2.1) is essentially Vandermonde (in the variables $r_1,r_2,\dots,r_p$), and hence is nonsingular. We note that $\mathfrak{X}$ is thin; cf. [27, Example 6.1].

Example 2.4. Suppose that $\mathfrak{X}$ is a dual polar scheme. Then, $c_i=(q^i-1)/(q-1)$ ($i=1,2,\dots,d$) for some prime power $q\geq 2$. Thus, the matrix (2.1) is again essentially Vandermonde (in the variables $q^{r_1},q^{r_2},\dots,q^{r_p}$), and hence is nonsingular. We note that $\mathfrak{X}$ is thin; cf. [27, Example 6.1]. In Appendix A, we show that Theorem 1.2 is valid for dual polar schemes.

Next, we move on to the Q-polynomial case.

Theorem 2.5. *Suppose that $\mathfrak{X}$ is Q-polynomial, and let $e,r_1,r_2,\dots,r_p$ be integers with $p-1\leq e\leq d$ and $0\leq r_1<r_2<\dots<r_p\leq d$. If every irreducible T-module W with dual endpoint at most e is dual thin, and satisfies $\rho^*(W)+\delta(W)\geq e$ and $|\{i \mid E^*_{r_i}W\neq 0\}|\geq \min\{p,e-\rho^*(W)+1\}$, then*

$$\dim\bigl(L_0(S)+L_1(S)+\dots+L_e(S)\bigr)=m_e+m_{e-1}+\dots+m_{e-p+1},$$

where $S=X_{r_1}\cup X_{r_2}\cup\dots\cup X_{r_p}$.

Proof. Again, fix a set $\mathcal{W}$ of irreducible T-modules in $\mathbb{C}^X$ such that $\mathbb{C}^X=\bigoplus_{W\in\mathcal{W}}W$. Observe that

$$L_j(X)=E_j\mathbb{C}^X=\bigoplus_{\substack{W\in\mathcal{W}\\ \rho^*(W)\leq j}}E_jW \quad (j=0,1,\dots,d),$$

so that

$$L_j(S)=\bigoplus_{\substack{W\in\mathcal{W}\\ \rho^*(W)\leq j}}\left(\sum_{i=1}^{p}E^*_{r_i}\right)E_jW \quad (j=0,1,\dots,d).$$

In particular, it follows that

$$\left(\sum_{j=0}^{e}L_j(S)\right)\cap W=\left(\sum_{i=1}^{p}E^*_{r_i}\right)\sum_{j=\rho^*(W)}^{e}E_jW\subseteq\bigoplus_{i=1}^{p}E^*_{r_i}W$$

for every $W\in\mathcal{W}$ with $\rho^*(W)\leq e$, and that

$$\sum_{j=0}^{e}L_j(S)=\bigoplus_{\substack{W\in\mathcal{W}\\ \rho^*(W)\leq e}}\left(\sum_{j=0}^{e}L_j(S)\right)\cap W.$$

Pick any $W\in\mathcal{W}$ with $\rho^*:=\rho^*(W)\leq e$, and let v be a nonzero vector in $E_{\rho^*}W$. First, suppose that $\rho^*\leq e-p+1$. Then, $v,A^*v,\dots,A^{*p-1}v\in\sum_{j=\rho^*}^{e}E_jW$. Since W is dual

thin, $\{E_i^* v \mid E_i^* W \neq 0\}$ is an orthogonal basis of W. Thus, the vectors $\big(\sum_{i=1}^p E_{r_i}^*\big) A^{*h} v = \sum_{i=1}^p \theta_{r_i}^{*h} E_{r_i}^* v$ $(h = 0, 1, \ldots, p-1)$ belong to $\big(\sum_{j=0}^e L_j(S)\big) \cap W$ and are linearly independent, since $E_{r_i}^* W \neq 0$ $(i = 1, 2, \ldots, p)$ and the coefficient matrix is Vandermonde. Thus, $\dim\big(\big(\sum_{j=0}^e L_j(S)\big) \cap W\big) = p$. Next, suppose that $e - p + 2 \leq \rho^* \leq e$. Likewise, we find that the vectors $\big(\sum_{i=1}^p E_{r_i}^*\big) A^{*h} v$ $(h = 0, 1, \ldots, e - \rho^*)$ belong to $\big(\sum_{j=0}^e L_j(S)\big) \cap W$ and are linearly independent, from which it follows that $\dim\big(\big(\sum_{j=0}^e L_j(S)\big) \cap W\big) = e - \rho^* + 1$. Thus, it follows that

$$\begin{aligned}
\dim\left(\sum_{j=0}^{e} L_j(S)\right) &= \sum_{\substack{W \in \mathcal{W} \\ \rho^*(W) \leq e}} \min\{p, e - \rho^*(W) + 1\} \\
&= \sum_{\substack{W \in \mathcal{W} \\ \rho^*(W) \leq e}} \sum_{j=e-p+1}^{e} \dim(E_j W) \\
&= \sum_{j=e-p+1}^{e} \dim(E_j \mathbb{C}^X) \\
&= \sum_{j=e-p+1}^{e} m_j,
\end{aligned}$$

as desired. □

Remark 2.6. In view of [7, Lemma 7.1], the assumption $\rho^*(W) + \delta(W) \geq e$ in Theorem 2.5 holds provided that $e \leq \lceil d/2 \rceil$.

Example 2.7. Consider a Hamming scheme $H(d, q)$. Then, $\rho(W) = \rho^*(W)$ for every irreducible T-module W; cf. [27, Example 6.1]. Thus, the assumption of Theorem 2.5 is satisfied provided that $e \leq r_1 < r_2 < \cdots < r_p \leq d - e$. Of course, in this case the conclusion also follows from Proposition 1.5, Theorem 2.1, and Example 2.3.

Example 2.8. Suppose that $\mathfrak{X}$ is P-polynomial, Q-polynomial, and bipartite. In this case, Caughman [7] showed that $\mathfrak{X}$ is thin, and that every irreducible T-module W satisfies $\delta(W) = d - 2\rho^*(W)$ and $\rho^*(W) \leq \rho(W) \leq 2\rho^*(W)$. Thus, the assumption of Theorem 2.5 is satisfied provided that $2e \leq r_1 < r_2 < \cdots < r_p \leq d - e$.

Example 2.9. Suppose that $\mathfrak{X}$ is a Johnson scheme $J(v, d)$. Then, $\mathfrak{X}$ is thin, and every irreducible T-module W satisfies $\rho(W) \leq \rho^*(W)$; cf. [27, Example 6.1]. Thus, the assumption of Theorem 2.5 is satisfied provided that $e \leq r_1 < r_2 < \cdots < r_p \leq d - e$.

3 A characterization of Hamming schemes

In this section, for $d \geq 6$, we characterize the Hamming schemes $H(d, q)$ as the formally self-dual P- and Q-polynomial association schemes with the property $\mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) = L_0(X) + L_1(X)$. We begin with the following result:

Proposition 3.1. *Suppose that $\mathfrak{X}$ is P-polynomial, Q-polynomial, and that* $\mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) = L_0(X) + L_1(X)$*. Then, $c_i/(\theta_i^* - \theta_0^*)$ is independent of $i = 1, 2, \ldots, d$.*

Proof. Since $A\hat{u}_0 \in E_1^*\mathbb{C}^X$, it follows that the vector

$$\left(\sum_{i=1}^{d} E_i^* A_{i-1} E_1^*\right) A\hat{u}_0 = \sum_{i=1}^{d} E_i^* A_{i-1} A\hat{u}_0 = \sum_{i=1}^{d} c_i A_i \hat{u}_0$$

belongs to $\mathrm{Hom}_1(X) \subset L_0(X) + L_1(X)$. On the other hand, this vector is in the primary T-module $\mathrm{span}\{\hat{u}_0, A_1\hat{u}_0, \ldots, A_d\hat{u}_0\} = \mathrm{span}\{E_0\hat{u}_0, E_1\hat{u}_0, \ldots, E_d\hat{u}_0\}$. Thus, it is written as

$$\sum_{i=1}^{d} c_i A_i \hat{u}_0 = \alpha E_0 \hat{u}_0 + \beta E_1 \hat{u}_0 = \frac{1}{|X|} \sum_{i=0}^{d} (\alpha + \beta\theta_i^*) A_i \hat{u}_0,$$

for some $\alpha, \beta \in \mathbb{C}$. Comparing the coefficients of $\hat{u}_0$, we find $\beta = -\alpha/\theta_0^*$, and hence

$$c_i = \frac{\alpha}{|X|\theta_0^*}(\theta_0^* - \theta_i^*) \quad (i = 1, 2, \ldots, d),$$

as desired.[3] □

Using this result, we now prove the following theorem:

Theorem 3.2. *Suppose that $\mathfrak{X}$ is formally self-dual, P-polynomial (and Q-polynomial), and satisfies* $\mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) = L_0(X) + L_1(X)$*. If $d \geq 6$, then $\mathfrak{X}$ is the Hamming scheme $H(d, q)$ for some q.*

Proof. Since $\mathfrak{X}$ is formally self-dual, in the notation of [5, Section 3.5] and [25, Section 2], the parameters of $\mathfrak{X}$ satisfies one of the following cases: (I) with $s = s^* \neq 0$; (I) with $s = s^* = 0$; (II) with $s = s^*$; (IIC); and (III) with $s = s^*$.[4]

First, consider Case (I) with $s = s^* \neq 0$. Then, it follows that

$$\frac{c_i}{\theta_i^* - \theta_0^*} = \frac{q^i(1 - sq^{i+d+1})(r_1 - sq^i)(r_2 - sq^i)}{sq^d(1 - sq^{i+1})(1 - sq^{2i})(1 - sq^{2i+1})} \quad (i = 1, 2, \ldots, d-1),$$

and this is independent of i by Proposition 3.1, so that

$$sq^d(1 - sq^{i+1})(1 - sq^{2i})(1 - sq^{2i+1}) = (\theta_1^* - \theta_0^*)q^i(1 - sq^{i+d+1})(r_1 - sq^i)(r_2 - sq^i)$$

for $i = 1, 2, \ldots, d-1$, and this identity is valid for $i = d$ as well. However, as polynomials in q^i, the left hand side is of degree five, whereas the right hand side is of degree four. Since $d \geq 6$, this is impossible. Case (I) with $s = s^* = 0$ is ruled out in the same way.

[3] In fact, we have $\alpha = \sum_{i=1}^{d} c_i k_i$.

[4] In the terminology of [28], these are of q-Racah, affine q-Krawtchouk, Racah, Krawtchouk and Bannai/Ito types, respectively.

Next, consider Case (II) with $s = s^*$. Then, it follows that

$$\frac{c_i}{\theta_i^* - \theta_0^*} = \frac{(i+s+d+1)(i+s-r_1)(i+s-r_2)}{(i+1+s)(2i+1+s)(2i+s)} \quad (i = 1, 2, \ldots, d-1).$$

Again, as polynomials in i, the denominator must be a scalar multiple of the numerator. In particular, they have the same roots. Since $1+s \neq s+d+1$, we may assume that $1+s = s-r_1$, i.e., $r_1 = -1$. Then, since $r_1 + r_2 = s + s^* + d + 1$, we have $r_2 = 2s + d + 2$. Using this and $\{s+d+1, s-r_2\} = \{(1+s)/2, s/2\}$, it follows that $d = \pm 1/4$, which is absurd.

If $\mathfrak{X}$ satisfies Case (III) with $s = s^*$, then by the classification due to Terwilliger [24], it follows that $\mathfrak{X}$ is isomorphic to $H(d, 2)$ (d even) or the bipartite half of $H(2d+1, 2)$, but with respect to the second P-polynomial orderings.[5] We have $c_i = i$ ($i = 1, 2, \ldots, d$) in either case, and it follows that $c_i/(\theta_i^* - \theta_0^*)$ cannot be constant, since $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are not an arithmetic progression.

Thus, we are left with Case (IIC). In this case, by the classification due to Egawa [15], $\mathfrak{X}$ is a Hamming scheme or a Doob scheme. If $\mathfrak{X}$ is a Hamming scheme, then we are done. Thus, suppose that $\mathfrak{X}$ is a Doob scheme. Then, there is a thin irreducible T-module W with $\rho(W) = 1$, $\rho^*(W) = 2$, and $\delta(W) = d - 2$. This fact follows from Tanabe's description [22] of the irreducible T-modules of Doob schemes, but we may also prove it as follows. The local graph of the Doob graph (X, R_1) (which has adjacency matrix $E_1^* A E_1^*$) is a disjoint union of hexagons and 3-cliques, so that it has -2 as an eigenvalue. On the other hand, we have $-1 - b_1/(1+\theta_1) = -2$. Thus, by a result of Go and Terwilliger [16, Theorem 9.8], any eigenvector (in $E_1^* \mathbb{C}^X$) of $E_1^* A E_1^*$ with eigenvalue -2 generates such a T-module. Now, let v be a nonzero vector in $E_1^* W$. Then, $\big(\sum_{i=1}^d E_i^* A_{i-1} E_1^*\big) v$ is nonzero and belongs to $\mathrm{Hom}_1(X)$. However, since $\rho^*(W) = 2$, it is contained in $L_2(X) + L_3(X) + \cdots + L_d(X)$. Thus, we conclude that $\mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) \neq L_0(X) + L_1(X)$, and the proof is complete. $\square$

A Comments on Theorem 1.2

In this appendix, we generalize Theorem 1.2 to dual polar schemes (Theorem A.6). Suppose that $\mathfrak{X}$ is a dual polar scheme, so that X is the set of maximal isotropic subspaces of a vector space V over a finite field, equipped with a non-degenerate form (alternating, Hermitian, or quadratic) of Witt index d. For convenience, we shall work with the dual polar graph (X, R_1) with path-length distance ∂.

Lemma A.1. *Let $x, y, z \in X$. Then, $\partial(x, z) + \partial(z, y) = \partial(x, y)$ if and only if $x \cap y \subseteq z = (x \cap z) + (y \cap z)$.*

Proof. Immediate from $\dim(x \cap z) + \dim(y \cap z) \leq d + \dim(x \cap y \cap z) \leq d + \dim(x \cap y)$. $\square$

[5] The second P-polynomial ordering of the Johnson scheme $J(2d+1, d)$ (corresponding to the Odd graph O_{d+1}) satisfies Case (III), but with $s = 2d+3$ and $s^* = 2d+2$.

For the moment, fix $x, y \in X$ and write $i = \partial(u_0, x)$, $j = \partial(u_0, y)$, $h = \partial(x, y)$, and $\ell = \dim(u_0 \cap U)$, where $U = x \cap y$. We note that $\ell \geq d - i - j$. Our goal is to show that $f_x f_y \in \mathrm{Hom}_{d-\ell}(X)$. We set $X' = \{z \in X \mid U \subseteq z\}$, and observe that X' induces a dual polar graph with diameter h.

Lemma A.2. *For every $z \in X$, there is a unique $z' \in X'$ such that $\partial(z, z') = \partial(z, X')$. Moreover, it holds that $\partial(z, z_1) = \partial(z, z') + \partial(z', z_1)$ for all $z_1 \in X'$.*

Proof. Set $z' = U + (z \cap U^{\perp}) \in X'$. Pick any $z_1 \in X'$. Then, $z' = (z \cap z') + (z_1 \cap z')$ since $U \subseteq z_1$ and $z \cap U^{\perp} \subseteq z$. Moreover, $z \cap z_1 \subseteq z \cap U^{\perp} \subseteq z'$. Thus, $\partial(z, z_1) = \partial(z, z') + \partial(z', z_1)$ by Lemma A.1, and the result follows. ☐

Lemma A.3. *Suppose that $z \in X'$ satisfies $f_x(z) = f_y(z) = 1$. Then, $\partial(u_0, z) = d - \ell$.*

Proof. First, $u_0 \cap z \subseteq x$ by Lemma A.1 and since $\partial(u_0, x) + \partial(x, z) = \partial(u_0, z)$. Likewise, $u_0 \cap z \subseteq y$. Thus, $u_0 \cap z \subseteq u_0 \cap U$. On the other hand, $u_0 \cap U \subseteq u_0 \cap z$. it follows that $u_0 \cap z = u_0 \cap U$, as desired. ☐

Lemma A.4. *For every $z \in X$ such that $f_x(z) = f_y(z) = 1$, there is a unique $z' \in X'$ such that $f_x(z') = f_y(z') = 1$ and $f_{z'}(z) = 1$.*

Proof. Let $z' (= U + (z \cap U^{\perp}))$ be as in Lemma A.2. Then, $f_x(z') = f_y(z') = 1$ and $f_{z'}(z) = 1$. To show the uniqueness, suppose that $z_1 \in X'$ satisfies $f_x(z_1) = f_y(z_1) = 1$ and $f_{z_1}(z) = 1$. Then, it follows from Lemma A.3 that $\partial(z, z') = \partial(z, z_1)$. But then, we must have $z' = z_1$ by Lemma A.2, and the proof is complete. ☐

Proposition A.5. *With the above notation, it holds that*

$$f_x f_y = \sum_{\substack{z \in X' \\ f_x(z) = f_y(z) = 1}} f_z \in \mathrm{Hom}_{d-\ell}(X).$$

Proof. Immediate from Lemmas A.3 and A.4. ☐

Theorem A.6. *Theorem 1.2 is valid for dual polar schemes.*

Proof. Suppose that $f \in \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_e(X)$ satisfies $f|_Y \equiv 0$. Then, $f^2 \in \mathrm{Hom}_0(X) + \mathrm{Hom}_1(X) + \cdots + \mathrm{Hom}_{2e}(X)$ by Proposition A.5. Thus,

$$\sum_{i=1}^{p} \frac{w(Y_{r_i})}{k_{r_i}} \sum_{x \in X_{r_i}} (f(x))^2 = \sum_{y \in Y} w(y)(f(y))^2 = 0,$$

from which it follows that the restriction map $\mathrm{Hom}_0(S) + \mathrm{Hom}_1(S) + \cdots + \mathrm{Hom}_e(S) \to \mathcal{F}(Y)$ $(f|_S \mapsto f|_Y)$ is injective, and the result follows by comparing the dimensions. ☐

B Comments on Proposition 1.5

We use the notation in the proof of Proposition 1.5. We mentioned there that the function f_z belongs to $\sum_{i=0}^{j} L_i(X)$. While this fact is just a special case of a more general result about regular semilattices [10, 11, 21], we now provide an independent proof.

We identify $\{0, 1, \ldots, q-1\}$ with the additive group $\mathbb{Z}/q\mathbb{Z}$. Let $\zeta \in \mathbb{C}$ be a primitive q^{th} root of unity. Then, the *additive group* X and its dual group X^* are isomorphic, and an isomorphism is given by $x = (x_1, x_2, \ldots, x_d) \mapsto \varepsilon_x$, where $\varepsilon_x(y) = \zeta^{\sum_{\ell=1}^{d} x_\ell y_\ell}$ for every $y = (y_1, y_2, \ldots, y_d) \in X$. In fact, it is well known (and is easily checked) that $L_i(X) = \operatorname{span}\{\varepsilon_x \mid x \in X_i\}$ (over $\mathbb{C}$) for $i = 0, 1, \ldots, d$, i.e., $H(d, q)$ is self-dual.

Assume that $i > j$, and pick any $y = (y_1, y_2, \ldots, y_d) \in X_i$. Then, the (standard) Hermitian inner product between ε_y and f_z is given by

$$\left(\prod_{h=1}^{j} \zeta^{z_{\ell_h} y_{\ell_h}} \right) \left(\prod_{\ell \neq \ell_1, \ldots, \ell_j} \left(\sum_{x_\ell = 0}^{q-1} \zeta^{x_\ell y_\ell} \right) \right).$$

Since $i > j$, there is an $\ell \neq \ell_1, \ldots, \ell_j$ such that $y_\ell \neq 0$. For this ℓ, we have $\sum_{x_\ell=0}^{q-1} \zeta^{x_\ell y_\ell} = 0$. Thus, f_z is orthogonal to ε_y. It follows that f_z is orthogonal to $\sum_{i=j+1}^{d} L_i(X)$, and hence it is contained in $\sum_{i=0}^{j} L_i(X)$, as desired.

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